Tests of General Relativity Using Starprobe Radio Metric Tracking Data

Kenneth D. Mease,* John D. Anderson,† Lincoln J. Wood,‡ and Lisa K. White§
Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California

The potential of a proposed spacecraft mission called Starprobe for testing general relativity and providing information on the interior structure and dynamics of the sun is investigated. The current mission plan is to place a spacecraft in a highly eccentric, highly inclined solar orbit with a perihelion distance of four solar radii. Parametric, gravitational perturbation terms are derived that represent relativistic effects and effects due to spatial and temporal variations in the solar potential at a given radial distance. The perturbation terms are incorporated into the equations of motion and radio metric data models for Starprobe. A covariance analysis based on Kalman filtering theory predicts the accuracies with which the free parameters in the perturbation terms can be estimated with radio metric tracking data through the process of trajectory reconstruction. It is concluded that Starprobe can contribute significant information on both the nature of gravitation and the structure and dynamics of the solar interior.

Introduction

SENDING a probe to the sun offers unique opportunities for testing general relativity and for refining models of the solar interior. One of the few opportunities for experimentally testing general relativity in our solar system is to verify the predicted perihelion advance of Mercury's orbit. Indeed, the advance has been measured to about 0.5%.[1] (All errors quoted in this paper are in terms of the standard deviation.) However, it has been apparent since the early 1960s that the advance could also be due, at least in part, to a solar oblateness.[2] Attempts to measure the oblateness by Earth-based visual methods have yielded conflicting results.[3,4] Moreover, it is not the visual oblateness, but the oblateness of the surfaces of constant gravitational potential, that is important in the perihelion advance of Mercury's orbit. In the standard spherical harmonic model of the external solar gravitational potential, it is the quadrupole moment coefficient J_2 that characterizes the amount of oblateness. The value of J_2 also places limits on the density, angular velocity, and magnetic field distributions within the sun and is therefore useful in developing and testing models of the solar interior.[5] Thus, a direct determination of the value of J_2 is important.

The standard method of determining a body's gravitational coefficients, such as J_2, is to infer their values from perturbations (deviations not predicted by Newtonian point-mass mechanics) in the trajectory of a probe flying close by or orbiting the body. With this motivation, along with the additional motivation for particles and fields studies as well as imaging science at the sun, a mission called Starprobe has been proposed and studied.[5,6] The most recent mission scenario is to place a spacecraft in a highly eccentric, highly inclined heliocentric orbit with a perihelion distance of four solar radii. Given a carefully designed Doppler tracking system and an onboard "drag-compensation" system to actively maintain the spacecraft on a trajectory determined almost exclusively by gravitational forces, it is possible,[7-12] to determine J_2 to a precision of one or two parts in 10^8. This would be a very useful result to both relativity testing and solar modeling.

The purpose of this paper is to examine additional scientific objectives that may be achievable through an analysis of gravitational perturbations on the Starprobe spacecraft and the radio signals used to track it. These objectives include looking for preferred-frame effects, testing a new theory of relativity proposed by Moffat,[13] determining the sun's angular momentum, measuring a normal mode of solar oscillation, and detecting a distortion in the surfaces of constant gravitational potential that has been hypothesized by Dicke.[14] The appropriate gravitational perturbations are represented parametrically in the equations of motion for the probe and in the models for the radio metric tracking data. Then, a covariance analysis, based on Kalman filtering theory, predicts the accuracies with which the unknown parameters characterizing the perturbations can be determined with Earth-based radio metric tracking data. The accuracies are computed as functions of the quality and quantity of the radio metric data, the parameters specifying the solar orbit, the magnitude of the effective nongravitational accelerations acting on the spacecraft, and the errors in the tracking station locations and in the ephemeris of the Earth relative to the sun. The results of the covariance analysis provide a quantitative measure of the scientific value of one aspect of the Starprobe mission and, in addition, useful insight into how to design the mission to maximize this scientific value. We wish to stress, however, that our analysis assumes that Starprobe tracking data will, in fact, conform to the parametric theories employed in our study. This may not be the case, which leads to the point that the ultimate scientific value of a data-gathering mission such as Starprobe can never be completely determined in advance. The data may well contain features that we have not anticipated.

Gravitational Perturbations and Their Parametric Representations

PPN Formalism

The parameterized post-Newtonian (PPN) formalism is a convenient framework in which to test theories of gravitation. This formalism[15] provides a general space-time metric for testing a broad class of metric theories of gravitation in the solar system. The PPN metric is a truncated expansion about the Minkowski "flat-space" metric in terms of dimensionless gravitational potentials of varying degrees of smallness. The

Presented as Paper 82-0205 at the AIAA 20th Aerospace Sciences Meeting, Orlando, Fla., Jan. 11-14, 1982; submitted Feb. 5, 1982; revision received Feb. 28, 1983.

*Member Technical Staff, Navigation Systems Section. Member AIAA.
†Senior Research Scientist, Systems Division.
‡Supervisor, Future Mission Studies Group, Navigation Systems Section. Senior Member AIAA.
§Engineer, Navigation Systems Section.

PPN metric includes 10 free parameters ($\beta,\gamma,\xi,\alpha_1,\alpha_2,\alpha_3,\zeta_1,\zeta_2,\zeta_3,\zeta_4$). The various metric theories are distinguished by the particular values assumed by these parameters.

For the Starprobe studies, we shall consider a reduced form of the PPN metric in which only the terms involving β,γ, and α_1 are retained. The terms involving β and γ represent the lowest-order terms in the post-Newtonian description of space and time and, hence, are essential to tests of general relativity. Heuristically speaking, β is a measure of the nonlinearity in the superposition law for gravity, and γ is a measure of the space-curvature produced by a unit rest mass. General relativity predicts a value of unity for both β and γ. The value of γ has been most accurately determined from the apparent time delay of electromagnetic radiation passing near the sun. The most recent analysis of the time-delay data for the Viking spacecraft near conjunction yields[16]

$$\gamma = 1.000 \pm 0.002 \tag{1}$$

The most accurate data for determining the value of β are measurements of the perihelion advance of Mercury's orbit. The measured excess advance, the amount not explained by Newtonian point-mass mechanics, as determined from planetary ranging data, is[1]

$$\Delta\omega = 43.''3 \pm 0.''2 \text{ per century} \tag{2}$$

In the PPN formalism, neglecting all terms contributing less than 0.''1 per century, the theoretical precession is

$$\Delta\omega = 42.''98(2+2\gamma-\beta)/3 + 1.''26 J_2 \times 10^5 - 124.''48\alpha_1 \tag{3}$$

Thus, although the perihelion advance is measured directly, its interpretation as a test of general relativity is confounded by the presence of too many unknown parameters. Independent measurements of J_2 and α_1 by Starprobe would provide a way out of this dilemma. The parameter α_1 is one of three preferred-frame parameters ($\alpha_1,\alpha_2,\alpha_3$), all of which are zero in general relativity, but which could take on nonzero values for gravitational theories based on a preferred frame of rest for the universe. We do not include α_2 and α_3 because rather stringent upper bounds have been set on their magnitudes by planetary motions[1] and by data on Earth tides.[17] On the other hand, the parameter α_1 could be as large as 0.1 and still go undetected in existing data. Starprobe could make a significant contribution to experimental gravitation by reducing the uncertainty in our knowledge of the value of α_1.

We shall also include a post-Newtonian term containing an additional unknown parameter τ in the metric for the Starprobe studies for the purpose of considering a recent theory of gravitation put forth by Moffat.[13] This theory is not represented in the standard PPN metric; and, in that regard, it must be considered separately from other competing theories of gravitation when performing experimental tests. In addition, the usual tests of general relativity do not place particularly strong limits on the validity of the Moffat theory. The parameter that Moffat uses in the formulation of his theory is ℓ, which is measured in units of distance and which assumes a unique value for each gravitating body. We use the dimensionless parameter $\tau = (\ell_s/R_s)^4$, where ℓ_s is the value of ℓ for the sun and R_s is the mean radius of the sun. The tightest bound on ℓ_s, derived from interplanetary radar observations of Mercury's perihelion advance,[18] is $|\ell_s| \le (2.92 \pm 0.10) \times 10^3$ km; however, the gravitational oblateness of the sun, as characterized by J_2, is ignored in the derivation of the bound. A less stringent bound, but one that does not depend on the value of J_2, is derived from the Viking time delay data, yielding $|\ell_s| \le 1.13 \times 10^4$ km.[18] Starprobe has the potential to measure ℓ_s, or equivalently τ, to an accuracy that would either limit the theory so severely that it becomes uninteresting as a competitor to general relativity (in which $\ell_s = 0$) or, alternatively, to show that it is preferable.

Equations of Motion

The equations of motion for the spacecraft, which are consistent with the PPN metric described above, are derived in the Appendix. The equations assume the convenient form of Newton's second law of motion, with additional perturbative acceleration terms characterizing the post-Newtonian effects. We include a further post-Newtonian effect, the Lense-Thirring effect, which represents the dragging of the inertial frame of the solar system by the rotating sun. The perturbative acceleration caused by the Lense-Thirring effect is given by Weinberg.[19] Because the effect is a function of the angular momentum of the sun, it is possible that Starprobe could measure this important quantity. With the addition of the Lense-Thirring term, the equations of motion, in vector form, are (dots denote derivatives with respect to coordinate time)

$$\begin{aligned}
\ddot{\boldsymbol{r}} = &-\frac{\mu}{r^3}\boldsymbol{r} \\
&+\frac{m}{r^3}\left\{\left[2(\beta+\gamma)\frac{\mu}{r} - \gamma(\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}})\right]\boldsymbol{r} + 2(1+\gamma)[\boldsymbol{r}\cdot\dot{\boldsymbol{r}}]\dot{\boldsymbol{r}}\right\} \\
&+\alpha_1\left(\frac{m}{2r^3}\right)\left\{[\boldsymbol{w}\cdot\dot{\boldsymbol{r}}]\boldsymbol{r} - [\boldsymbol{r}\cdot\dot{\boldsymbol{r}}]\boldsymbol{w}\right\} \\
&-\tau\frac{R_s^4}{r^6}\left\{\left[-2c^2 + 13\left(\frac{\mu}{r}\right) - (\dot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}})\right]\boldsymbol{r} + 2[\boldsymbol{r}\cdot\dot{\boldsymbol{r}}]\dot{\boldsymbol{r}}\right\} \\
&+L\left(\frac{2m}{r^3}\right)\left\{\frac{3}{r^2}[\boldsymbol{r}\times\dot{\boldsymbol{r}}][\boldsymbol{r}\cdot\boldsymbol{i}_z] + [\dot{\boldsymbol{r}}\times\boldsymbol{i}_z]\right\}
\end{aligned} \tag{4}$$

where $\boldsymbol{r}$, $\dot{\boldsymbol{r}}$, and $\ddot{\boldsymbol{r}}$ are, respectively, the heliocentric position, velocity, and acceleration vectors of the spacecraft, and r is the Euclidean norm of $\boldsymbol{r}$, μ the product of the gravitational constant and the mass of the sun, c the velocity of light in vacuo, $m = \mu/c^2$ one-half the gravitational radius of the sun, R_s the mean radius of the sun, $\boldsymbol{i}_z$ a unit vector parallel to the spin axis of the sun, and L the magnitude of the sun's angular momentum per unit mass (the specific angular momentum vector is $L\boldsymbol{i}_z$, i.e., we assume that a principal axis and the spin axis spatially coincide). A dot between two vectors denotes an inner product, and an $\times$ between two vectors denotes a cross product. The velocity vector $\boldsymbol{w}$ represents the motion of the solar system with respect to the frame of rest of the universe. We assume that $\boldsymbol{w}$, in the Earth mean equator and equinox of 1950 coordinate frame, is a constant vector equal to (−353.44, 28.93, 34.08) km/s, a weighted average of the four most recent measurements of the velocity of the solar system with respect to the cosmic microwave background radiation.[20]

The first term on the right-hand side of Eq. (4) represents the Newtonian acceleration imposed on the spacecraft by the sun and is equal to $-\nabla V$, where $V = -\mu/r$ is the gravitational potential and ∇ the gradient operator. This model of the potential is exact only if the sun acts gravitationally as a spherically symmetric distribution of mass. However, with a perihelion distance of four solar radii, the Starprobe spacecraft will be affected by asymmetries in the solar mass distribution, constant or time varying. Detection of these mass distribution asymmetries can yield important information on the internal structure and dynamics of the sun. Consequently, we shall henceforth model the potential as

$$\begin{aligned}
V = &-\frac{\mu}{r} + \left[J_2 + A\sin(2\pi f t + \Psi)\right]\frac{\mu}{r}\left(\frac{R_s}{r}\right)^2 P_2(\sin\phi) \\
&-\frac{\mu}{r}\left(\frac{R_s}{r}\right)^2\left[(C_{21}\cos\lambda + S_{21}\sin\lambda)P_2^1(\sin\phi) + (C_{22}\cos 2\lambda\right. \\
&\left.+ S_{22}\sin 2\lambda)P_2^2(\sin\phi)\right] + J_4\frac{\mu}{r}\left(\frac{R_s}{r}\right)^4 P_4(\sin\phi)
\end{aligned} \tag{5}$$

where ϕ is the latitude measured from the solar equator, λ the longitude measured from the prime meridian, P_n the nth order Legendre polynomial, and P_n^m the associated Legendre function of the first kind. The first term on the right-hand side has been described above. It is the lowest-order term, and, through most of Starprobe's heliocentric orbit, it represents the solar potential very well. However, for a few hours around perihelion, the second, third, and fourth terms may be required. The second term is due to the quadrupole moment, which characterizes the polar flattening and the equatorial bulge of the surfaces of constant potential. The constant parameter J_2 measures the mean level of the oblateness, while the term $A\sin(2\pi ft+\Psi)$ measures a sinusoidal fluctuation about the mean level as a function of time. The values of J_2, predicted by various theoretical models of the solar interior, range from 9×10^{-8} to 2×10^{-6} (see Ref. 5). The value of J_2 has also been inferred from measurements of the visual oblateness of the sun. Dicke and Goldenberg[3] infer a value of $(2.47\pm0.23)\times10^{-5}$ while Hill and Stebbins[4] infer a value of $(0.10\pm0.43)\times10^{-5}$. More reently, a value of $(0.55\pm0.13)\times10^{-5}$ has been inferred from the rotational splitting of global solar oscillations.[38]

A new method of probing the solar interior involves the detection and identification of the normal modes of global solar oscillations. The oscillations are external manifestations of internal processes and can be employed in a manner analogous to the way in which seismic waves are used to infer the internal structure and dynamics of the Earth. The existence of 5 min period global oscillations is now generally accepted.[21] Oscillations with a 160 min period have also been reported[22,23]; however, their existence is a point of controversy. Thus far, the evidence for oscillations has come from Earth-based visual or radio frequency Doppler measurements of the sun's surface. The interpretation of such measurements is difficult, because a number of other phenomena (including artifacts of the data analysis) can produce or imply oscillations that can be confused with the oscillations due to internal processes. Therefore, it has been suggested[24,25] that one could alternatively look for perturbations in the solar gravitational potential, which are predicted to occur as a result of certain modes of the global oscillations. The modification $A\sin(2\pi ft+\Psi)$, which has been added to the constant parameter J_2 in the second term of Eq. (5), represents a time-varying perturbation to the quadrupole moment. Christensen-Dalsgaard and Gough[26] have suggested that the reported 160 min oscillations might give rise to an oscillatory quadrupole moment. Assuming a uniform solar rotation, the amplitude of the quadrupole moment oscillation would be only a factor of three or so less than the static component, $J_2=2\times10^{-7}$, induced by centrifugal forces.[26] For the Starprobe studies, we shall let $f=1/(160\text{ min})$ and assume that Ψ is known; thus, the amplitude A is the only free parameter characterizing the oscillation. Note that Ψ is, in fact, not known. We are assuming here that concurrent Earth-based observations will yield the value of Ψ. For our baseline case, $\Psi=0$; however, if it is set equal to some other value, the covariance results do not change (based on numerical experiments).

The rationale for the third term on the right-hand side of Eq. (5) is as follows. Analyzing optical measurements of the elliptical figure of the sun made in 1966, Dicke found a periodic signal that he hypothesized to be caused by a solar distortion rotating rigidly as a wave on the surface.[14] The distortion is in the form of an ellipsoid, with its major axis tilted 5 deg with respect to the equatorial plane. The sidereal rotation period is 12.38 days,[27] about half that of the surface rotation at the equator. Both the distortion and the surface rotate about a common spin axis. Dicke attributed the rotating distortion to a distorted core, due most likely to a strong ($\approx10^8$ G) magnetic field, "frozen" in the core.[14] If the core is indeed distorted, it will induce distortions in the surfaces of constant gravitational potential. To a first approximation, the distortions can be modeled by including the tesseral harmonic terms that comprise the third term in the potential of Eq. (5). The parameters C_{21}, S_{21}, C_{22}, and S_{22} can be treated as static by allowing the gravitational potential model to rotate about the sun's spin axis with a sidereal period of 12.38 days.

The fourth term in the potential is included for completeness; however, it has been shown[12] that, even if the magnitude of J_4 is equal to the largest theoretically determined upper limit of 1.5×10^{-8} (see Ref. 28), J_4 could not be determined by Starprobe.

Radio Metric Data Models

The coordinate time of propagation of a radio signal between a tracking station on Earth and Starprobe is

$$c\Delta t=\rho+(1+\gamma)m\,\ell n\left[\frac{r_E+r_P+\rho}{r_E+r_P-\rho}\right] -\frac{\tau R_s^4}{4r_E^3\sin^3\theta_1}\left[\theta_2-\frac{1}{2}\sin 2(\theta_1+\theta_2)+\frac{1}{2}\sin 2\theta_1\right] \qquad (6)$$

where ρ, r_E, and r_P are coordinate distances representing, respectively, the distance between the station and the spacecraft, the heliocentric distance of the station, and the heliocentric distance of the spacecraft. The angles θ_1 and θ_2 are, respectively, the sun/station/probe angle and the station/sun/probe angle. A derivation of Eq. (6) is presented in the Appendix. The second term on the right-hand side of the equation results from an "excess" relativistic time delay of the radio signal consistent with the description of space and time given by the PPN metric. The third term results from a decrease in the time delay predicted by the Moffat theory.

For a model of actual range data, the coordinate time delay of Eq. (6) must be transformed to a proper time for a clock at the tracking station on Earth, and the round-trip light-time equation solved. For a statistical error analysis, it is unnecessary to transform to proper time, because the parameters β, γ, α_1, and τ do not enter into the time transformation to the first order in m/r. Also, for an error analysis, the round-trip light time between a station on Earth and Starprobe can be represented by simply doubling Eq. (6).

In constructing actual round-trip Doppler data over a finite count time, the standard procedure is to compute round-trip range at the beginning and end of the counting interval and then to difference the two range computations to obtain the integrated Doppler. For the Starprobe studies, however, we use range rate to represent the Doppler data. These pseudo Doppler data are represented by differentiating Eq. (6) with respect to time.

Parameter Estimation

In the previous section, the anticipated gravitational perturbations to the Starprobe trajectory and the radio tracking signals have been characterized by parametric expressions. Provided that the expressions are sufficient to model the perturbations, what remains for Starprobe is to determine the values of the free parameters. The procedure for doing so can be cast as a standard orbit determination problem. Specifically, the equations of motion for the Starprobe spacecraft, coupled with the models of the radio metri data, are used to generate predicted tracking data as functions of the free parameters discussed above and other more conventional orbit determination parameters, such as the six components of the spacecraft state. The values of the parameters are then adjusted until the predicted data agree with the actual measured data in a weighted least-squares sense.

There are two technical problems that confront the parameter estimation. The first problem is that nongravitational forces such as solar radiation pressure, which cannot be modeled or measured to sufficient accuracy, will act on the spacecraft in addition to the gravitational forces discussed above. The solution to this problem is an onboard "drag-compensation" system.[29] The second problem is that the radio metric data around perihelion, which are crucial for estimating the parameters of interest, are corrupted by turbulence in the solar corona. A sophisticated Doppler tracking system is being studied to solve this problem.[30]

Covariance Analysis

A covariance analysis is performed in order to predict the accuracies with which the values of the parameters associated with gravitational perturbations can be determined with Starprobe radio tracking data and to determine how other parameters in the problem, and the uncertainties in their values, influence those accuracies. The parameters to be estimated and the radio metric data are assumed to be Gaussian random variables and, consequently, to be fully characterized by their mean values and covariance matrix. In the linearized Kalman filtering approach to parameter estimation, the time propagation and measurement updating of the covariance matrix is independent of the processing of the data residuals to compute the estimated mean values (although the converse is not true). Thus, given a data schedule, the statistical properties of the process and measurement noises, an a priori covariance matrix for the estimated parameters, and appropriate transition matrices based on a nominal trajectory, the a posteriori covariance matrix for the estimated parameters can be derived. The a posteriori covariance matrix is computed using Bierman's factorized formulation of the Kalman sequential filter.[31] Partial derivatives of the spacecraft state with respect to the estimated parameters are required for the linearized Kalman filter. They are obtained by integrating variational equations. The variational equations are formulated by differentiating the equations of motion with respect to the estimated parameters.

Due to the complicated nature of the computational software used to perform the covariance analysis, it is desirable to have an independent check of the results. The software used for the present analysis has been checked against another set of software developed independently by Anderson and Lau.[9] Under conditions that overlap as much as possible, the two sets of software produce essentially identical results.

Since the covariance analysis involves parametric studies of the sensitivity of the a posteriori error covariance matrix for the estimated parameters to variations in the a priori assumptions, it is useful to define a baseline case and then to vary certain a priori assumptions one at a time.

Baseline Case

The nominal spacecraft trajectory is consistent with the ΔV-EJGA design presented in detail in Ref. 32. Following gravity assists from the Earth and, subsequently, Jupiter, the spacecraft will be in a heliocentric orbit specified by the classical elements ($a = 406432329.0$ km, $e = 0.99331$, $i = 90.0$ deg, $\Omega = 160.14$ deg, $\omega = -178.66$ deg, $t_p = 7/16/94/0$ h) as referenced to the Earth mean ecliptic and equinox of 1950 coordinate frame. The time of perihelion given is consistent with a launch on Sept. 6, 1988. (This launch date is purely hypothetical.) The distance from the sun's center at perihelion is four solar radii. The heliocentric orbit is nearly polar. The transit time from pole to pole, through perihelion, is less than 14 h, the spacecraft reaching a speed of approximately 300 km/s at perihelion.

The radio tracking data arc extends from -7 to $+2$ days relative to perihelion. Both range and Doppler data types are assumed with standard deviations of 15 m and 0.1 mm/s (for a 1 min count time), respectively. Data rates are highest during the 10 h period centered at perihelion time: one datum every 30 min for range and one datum every 5 min for Doppler. See Table 2 of Ref. 12 for a detailed data schedule.

The estimated parameters are the heliocentric position and velocity components of the spacecraft and the Earth, J_2, A, β, γ, α_I, and τ and the effective nongravitational accelerations (i.e., the nongravitational accelerations that are not compensated for by the drag compensation system). The effective nongravitational accelerations are modeled as biases. The parameters ATAR, ATAX, and ATAY are the magnitudes of three such orthogonal accelerations, acting along the spacecraft roll, pitch, and yaw axes, respectively. (The roll axis is always oriented in the direction of the sun.) The effects of nongravitational accelerations, with finite correlation times, on the accuracy with which a somewhat reduced set of the above parameters can be estimated, have been reported elsewhere.[12]

Equivalent station location errors (for stations 14, 43, and 63) and errors in the parameters J_4 and μ are "considered" in the covariance analysis. Equivalent station location errors include crust-fixed station location errors plus residual calibration errors associated with transmission media effects, polar motion, and Earth spin rate, since all of these errors affect the estimation accuracy in roughly the same manner. To consider errors in the present context means to introduce the errors after the a posteriori covariance matrix has been computed by the filter. The filter is run first, sequentially processing all of the data, with the values of the considered parameters assumed to be known exactly. This yields a covariance matrix for the errors in the estimated parameters. Then, the covariance matrix for the errors in the estimated parameters, which are due only to the uncertainty in the considered parameters, is computed. The sum of this matrix and the covariance matrix produced by the filter yields the considered covariance matrix. It is important to distinguish between this approach to estimation in the presence of uncertain parameters[31] and the less conservative Schmidt-Kalman filtering approach.[33]

A priori standard deviations for the estimated and considered parameters are given in Table 1. The coordinates are referenced to the Earth mean-equator and equinox of 1950 frame. The errors in the components of the Earth state are correlated. The a priori standard deviations for the spacecraft state are chosen to be large, so as to have little impact on the final results. The a priori standard deviation for τ is equivalent to a conservative upper bound of $|\ell_s| \leq 2\times 10^4$ km. The a priori standard deviation for μ corresponds to a 20 m uncertainty in the astronomical unit. The a priori standard deviation for J_4 is equal to the largest theoretically derived upper bound on J_4.[28]

Also presented in Table 1 are the a posteriori standard deviations of the estimated parameters after the sequential processing of all of the data. The first column of a posteriori standard deviations contains the values computed by the filter, assuming no J_4, μ, or equivalent station location errors. The second column contains the "postfiltering" adjusted values that result when J_4, μ, and equivalent station location errors are considered. Observe that, for certain estimated constant parameters, the considered a posteriori standard deviation is larger than the corresponding a priori value. This situation arises when data appearing to contain information are used by the filter to reduce the uncertainty in the estimated parameters; yet, when the considered errors are accounted for, it turns out that the data did not contain truly useful information due to mismodeling effects and actually increased the uncertainty in the estimated parameter values. If only the μ and J_4 errors are considered, we find that the considered a posteriori standard deviations equal the a posteriori values from the filter. Thus, it is the equivalent station location errors that cause the disruptive effect. The rationale behind considering equivalent station location errors is that the alternative of estimating these quantities may lead to an unrealistic reduction in their uncertainties. In other words, the models in the covariance analysis software for equivalent station location parameters are not adequate to assess whether or not the associated errors could be reduced by estimation. However, it should be noted that the considered error analysis is a "worst-case" analysis in the sense that the filter is formulated in complete ignorance of the considered errors. Nonetheless, we choose to be conservative rather than overly optimistic and shall discuss only the considered a posteriori standard deviations in the remainder of the paper.

The baseline case was also run with an extended data arc, which began 30 days prior to perihelion and ended 30 days after perihelion. The a posteriori standard deviations did not differ significantly from those for the 9 day data arc. In fact, an examination of the evolution of the a posteriori standard deviations as each data point is processed reveals that the parameters characterizing the gravitational perturbations are not observable until roughly 10 h prior to perihelion. We have chosen to start the baseline data arc at 7 days before perihelion to assure that a realistic covariance for the spacecraft state has been established by the time the gravitational parameters become observable.

When the magnitude of the solar angular momentum L is included as an estimated parameter, we find that it is unobservable with Starprobe tracking. This is true even with an order of magnitude improvement in the Doppler accuracy ($\sigma = 0.01$ mm/s).

Table 1 Baseline standard deviations

Estimated parameter	Standard deviations A priori	A posteriori	Considered a posteriori
Spacecraft state			
x, km	1000.0	1.0	4.5
y, km	1000.0	1.0	3.0
z, km	1000.0	0.8	5.6
$\dot{x}$, mm/s	1000.0	2.4	9.4
$\dot{y}$, mm/s	1000.0	3.8	12.4
$\dot{z}$, mm/s	1000.0	4.9	29.9
Earth state			
x, km	4.7	1.2	9.6
y, km	4.7	0.7	10.7
z, km	9.3	2.1	25.1
$\dot{x}$, mm/s	1.0	0.2	1.3
$\dot{y}$, mm/s	1.0	0.8	1.7
$\dot{z}$, mm/s	2.0	1.8	5.0
$J_2 \times 10^8$	100.0	1.0	2.5
$A \times 10^8$	100.0	1.0	2.1
$\beta \times 10^2$	1.0	1.0	1.0
$\gamma \times 10^2$	1.0	1.0	1.6
$\alpha_l \times 10^3$	100.0	1.3	7.0
$\tau \times 10^{12}$	6.82×10^5	1.4	2.6
Effective nongravitational accelerations, km/s$^2 \times 10^{12}$			
ATAR	1.0	0.7	1.9
ATAX	1.0	1.0	1.2
ATAY	1.0	1.0	1.1
Considered parameter			
$J_4 \times 10^8$	1.5		
μ, km^3/s^2	68.0		
Station spin radii, m	0.8		
Station longitudes, m	0.8		
Station heights from equatorial plane, m	0.8		

Sensitivity Studies

The sensitivity of the above results to variations in certain baseline assumptions is shown in Tables 2-4. In each table, the conditions assumed to generate the results are identical to those of the baseline case, except as noted. Attention is restricted to the parameters of scientific interest. Only the considered a posteriori standard deviations, after the processing of all the data in the 9 day arc, are presented.

The motivation for generating Table 2 is that 1) an improved Doppler system is being discussed, possibly producing measurements with a standard deviation of 0.01 mm/s for a 1 min count time; and 2) it is of much interest to know whether the perihelion distance can be raised in order to reduce the heat shield requirements and thereby cut mission costs. A reduced value for station location errors is used for the more precise Doppler cases because the improved Doppler is useful only if there is a corresponding improvement in station location errors.[12] It is clear, from the results shown, that raising the perihelion distance from four solar radii quickly degrades the determination of the gravitational parameters. The apparent improvement seen for γ in the baseline Doppler accuracy (0.1 mm/s) case is an artifact of the considered statistics. The a priori error for γ is 1.0×10^{-2}, and a posteriori errors greater than this mean simply that the parameter is unobservable. For the improved Doppler system ($\sigma = 0.01$ mm/s), there is an improvement in the determination of some of the parameters and a reduction in the sensitivity to perihelion distance for J_2, A, and α_l. However, these results rest on the assumption of reduced station location errors.

In order to emphasize the importance of station location errors in the estimation problem, the baseline case has been run with four sets of station location errors (Table 3). The largest errors shown are the current values. The 0.8 m errors are projected errors for the 1990s.

The second half of Table 3 shows the sensitivity of the estimation accuracies to variations in the level of nongravitational acceleration biases. At the levels tested, these are less important error sources than the station locations. For a more comprehensive treatment of nongravitational accelerations, see Ref. 12.

We examined the sensitivity of the estimation accuracy to the inclination of the heliocentric orbit. The considered a posteriori standard deviation for J_2 goes from 2.5 to 3.4 to 19.0 parts in 10^8 as the inclination goes from 90 to 50 to 0 deg.

Table 2 Sensitivity to variation in perihelion distance R_p for two combinations of Doppler accuracy and station location errors

	Considered a posteriori standard deviations					
	Doppler accuracy (1σ) = 0.1 mm/s, Station errors[a] (1σ) = 0.8 m			Doppler accuracy = 0.01 mm/s, Station errors[a] = 0.2 m		
Estimated parameter	$R_p = 4R_s$	$6R_s$	$8R_s$	$4R_s$	$6R_s$	$8R_s$
$J_2 \times 10^8$	2.5	12.2	19.0	2.5	7.0	12.2
$A \times 10^8$	2.1	4.6	10.2	0.4	0.7	1.8
$\beta \times 10^2$	1.0	1.0	1.0	2.8	3.5	3.1
$\gamma \times 10^2$	1.6	1.2	1.1	5.8	7.6	5.8
$\alpha_l \times 10^3$	7.0	18.0	38.6	2.8	5.3	11.9
$\tau \times 10^{12}$	2.6	10.5	33.1	6.3	30.8	50.9

[a]Same for all three components.

Table 3 Sensitivity to variations in station location errors and to variations in the level of nongravitational acceleration biases

Changes from baseline assumptions (standard deviations of parameters)								
Station spin radii, m	Station longitudes, m	Station heights from equatorial plane, m	$J_2 \times 10^8$	Considered a posteriori standard deviations $A \times 10^8$	$\beta \times 10^2$	$\gamma \times 10^2$	$\alpha_1 \times 10^3$	$\tau \times 10^{12}$
0.0	0.0	0.0	1.1	1.0	1.0	1.0	1.3	1.4
0.2	0.2	0.2	1.0	1.1	1.0	1.0	2.1	1.5
0.8	0.8	0.8	2.5	2.1	1.0	1.6	7.0	2.6
1.0	2.0[a]	10.0	5.9	8.2	2.1	10.8	65.9	13.8
ATAR, km/s^2	ATAX, km/s^2	ATAY, km/s^2						
10^{-13}	10^{-13}	10^{-13}	2.1	2.1	1.0	1.6	6.5	2.4
10^{-12}	10^{-12}	10^{-12}	2.5	2.1	1.0	1.6	7.0	2.6
10^{-11}	10^{-11}	10^{-11}	3.0	2.1	1.0	1.7	7.9	2.9

[a]The longitude errors for the three stations are correlated ($\rho = 0.9$).

Table 4 Sensitivity to distortion in the solar gravitational potential as hypothesized by Dicke

	Standard deviation: tilt of major axis of solar distortion					
	0 deg		5 deg		10 deg	
Estimated parameter	A priori	Considered a posteriori	A priori	Considered a posteriori	A priori	Considered a posteriori
$J_2 \times 10^8$	100.0	2.5	100.0	4.3	100.0	7.3
$C_{21} \times 10^8$	0.0	0.0	35.0	30.7	68.0	34.3
$S_{21} \times 10^8$	0.0	0.0	35.0	3.0	68.0	2.9
$C_{22} \times 10^8$	0.0	0.0	0.75	0.75	3.0	3.2
$S_{22} \times 10^8$	0.0	0.0	0.75	0.83	3.0	5.6
$A \times 10^8$	100.0	2.1	100.0	1.9	100.0	1.8
$\beta \times 10^2$	1.0	1.0	1.0	1.0	1.0	1.0
$\gamma \times 10^2$	1.0	1.6	1.0	1.5	1.0	1.5
$\alpha_1 \times 10^3$	100.0	7.0	100.0	35.1	100.0	37.2
$\tau \times 10^{12}$	6.82×10^5	2.6	6.82×10^5	2.2	6.82×10^5	2.1

Table 5 Combination of the Starprobe results with the results of other relativity experiments

	Standard deviations	
Parameters	Starprobe alone	Starprobe plus three existing measurements
J_2	7.4×10^{-8}	0.93×10^{-8}
β	1.00	0.0024
γ	1.00	0.0007
α_1	0.0083	0.0006

Similar results are found for A, α_1, and τ. The parameters β and γ are unobservable at all three inclinations.

Table 4 represents an attempt to assess the impact of the distortion in the solar gravitational potential, which has been hypothesized by Dicke, on the parameter estimation problem. Our approach makes use of analytical expressions for transforming known gravity potential coefficients (J_2, C_{21}, S_{21}, C_{22}, S_{22}) in a given coordinate system to their corresponding values in a rotated coordinate system.[34] We begin by assuming $J_2 = 4.0 \times 10^{-6}$, a reasonable upper bound on the oblateness, and $C_{21} = S_{21} = C_{22} = S_{22} = 0$. These values are then transformed by either a 5 or 10 deg rotation about an axis in the solar equatorial plane that intersects the center of the sun and a given line of constant longitude. This produces nonzero values for C_{21}, S_{21}, C_{22}, and S_{22}. This procedure is repeated for different axes of rotation that vary in the longitude line they intersect (0-90 deg). This establishes the upper bounds on the four parameters, which are, in turn, used as a priori standard deviations. In effect, we are assuming that the distortion in the gravitational potential is describable by J_2 alone, but that we have chosen the wrong axis of symmetry for our gravity model (namely, the spin axis) and must therefore estimate all five coefficients (J_2, C_{21}, S_{21}, C_{22}, S_{22}), in order to characterize the distortion. In Table 4, the 0 deg case corresponds to the situation in which the correct axis of symmetry is the spin axis or, equivalently, in which the major axis of the ellipsoidal solar distortion is in the equatorial plane (i.e., untilted). While there are many alternative approaches to studying this problem, and our study is far from exhaustive, a few conclusions can be drawn. First, an examination of the full a posteriori covariance matrix reveals that C_{21} and S_{21} are correlated with α_1 (correlation coefficient = 0.97) and cause the disruption in its determination. Second, the determination of J_2 worsens as the tilt increases. Third, the inclination of the Starprobe orbit and the position of the sun in its rotation become important when the longitudinally dependent potential terms characterized by (C_{21}, S_{21}, C_{22}, S_{22}) are necessary to describe the potential. In the particular situation investigated here, the data contain information on S_{21} but not on C_{21}, C_{22}, or S_{22}.

Impact of Starprobe on Testing of Gravitational Theories

The overall impact of the Starprobe results on testing of gravitational theories becomes apparent when the predicted results are combined with existing measurements. Here we consider only the parameters J_2, β, γ, and α_1. There are three existing measurements that yield information about the four parameters. These are the measurement of Mercury's perihelion advance, the Viking time-delay measurement, and the measurement by laser ranging of the synodic perturbation in the orbit of the moon. Mercury's perihelion advance yields information on a linear combination of the four parameters [see Eq. (3)]. The Viking experiment measures γ alone [see

Eq. (1)]. The lunar laser data have been analyzed by two independent teams of investigators.[35,36] Similar results were obtained. We quote, here, the result with the larger error,[35] namely,

$$4\beta-\gamma-3-\alpha_I=0.00\pm0.03 \tag{7}$$

Thus, the parameter γ is well known from the Viking experiment, but there are no experiments at comparable accuracy for the other three parameters. The three parameters cannot be uniquely separated by only two measurements. In the past, whenever results have been quoted for β, it has been necessary to make the arbitrary assumption that $\alpha_I=0$. With the addition of the Starprobe measurement, this situation would be rectified.

To quantify this last statement, we have combined statistically the three existing measurements and the information from the Starprobe covariance analysis. Because all three existing measurements should be improved in the next few years, the current published errors have been reduced by a factor of three, a reasonable estimate. The resulting errors from Starprobe alone and from a combination of Starprobe with the three existing measurements are shown in Table 5. Note that the results given for Starprobe alone differ from those given for the baseline case in Table 1. This is because we have recomputed the considered a posteriori standard deviations for a relaxed set of a priori standard deviations for β and γ, both equal to unity, so as not to assume any prior knowledge of their values and thereby to isolate the Starprobe relativity experiment from all others.

Combining the Starprobe measurement with the three existing measurements reveals the true value of Starprobe for testing theories of gravitation. The addition of the fourth measurement not only allows the values of the four parameters to be determined, but both β and α_I are determined to an accuracy comparable to the accuracy of the Viking measurement of γ. Moreover, despite the degradation of the Starprobe determination of J_2 when the larger a priori errors for β and γ are assumed, the combined results give J_2 to one part in 10^8. While we shall not discuss the impact of these determinations in terms of particular theories of gravitation, it is safe to say that knowing the values of J_2, β, γ, and α_I to the predicted accuracies would reduce the number of viable theories.

In combining the Starprobe information with existing measurements, we have neglected the possibility of new future measurements that would provide another independent equation in the four parameters. Extensive tracking of the Viking lander into the 1990s or the tracking of Venus by means of an orbiter equipped with a ranging transponder could provide such an independent equation, and consequently reduce the dramatic contribution of Starprobe to a determination of β. However, in the absence of new types of measurements, the contribution of Starprobe is impressive and, even with new measurements, we would expect its contribution to be significant.

Conclusions

Under the assumptions of this analysis, we find that gravitational science objectives, in addition to an accurate determination of J_2, are achievable by Starprobe. The parameterized post-Newtonian, preferred-frame parameter α_I can be determined to seven parts in 10^3, more than an order of magnitude improvement over our current knowledge of its value. The Moffat parameter τ can be determined to 2.6 parts in 10^{12}. This corresponds to an upper bound of $|\ell_s|\leq 884$ km.

If a 160 min period quadrupole moment oscillation exists for the sun, its amplitude A can be determined to about two parts in 10^8. For the baseline case, the phase angle Ψ [see Eq. (5)] was set equal to zero. For other phase angles, the accuracy with which A is determined did not vary.

The ability of Starprobe, with the nominal orbital inclination of 90 deg with respect to the ecliptic plane, to detect the distortion hypothesized by Dicke or, in general, longitudinal variations in the solar gravitational potential, is limited. Although the orbit is not quite polar, since the solar equatorial plane is inclined 7.25 deg with respect to the ecliptic plane, the range of longitudes covered is rather small due to the short duration of the flyby (~20 h) and the long rotation period of the sun (~25 days). Nonetheless, Starprobe can provide some information on the pair of gravitational potential parameters (C_{21},S_{21}). Whether this information concerns just one of the parameters or a combination of the two depends on the specific longitudes probed by the spacecraft. The parameters C_{22} and S_{22} are not observable, assuming that the nominal trajectory is flown. Obviously, lower inclination orbits would be preferable for detecting longitudinal variations in the potential; however, for these orbits, the ability to determine J_2 is compromised. An intermediate inclination may be the answer.

The magnitude of the specific angular momentum L is not observable, under the assumptions of this analysis. This is true even with an order of magnitude improvement over the assumed baseline Doppler data accuracy.

The parameter J_2 can be determined to 2.5 parts in 10^8. In a previous covariance analysis,[12] which did not include the parameters α_I, τ, A, C_{21}, S_{21}, C_{22}, or S_{22} in any manner, it was found that J_2 could be determined to 1.6 parts in 10^8. Thus, the inclusion of the additional parameters degrades the J_2 determination only slightly. The parameters β and γ are not observable in the present analysis. In the previous analysis, β was unobservable, but the uncertainty in γ was reduced from its assumed a priori level. However, the resulting determination was not at all competitive with that by the Viking time-delay experiment.

Finally, combining the expected information from the Starprobe mission on the parameters J_2, β, γ, and α_I with that from three existing measurements (Mercury's perihelion advance, the Viking time delay, and the synodic perturbtion of the lunar orbit), we find that the values of the four parameters can be determined to accuracies sufficient to reduce the number of viable gravitational theories.

Appendix

In isotropic Cartesian coordinates, the gravitational field external to a spherical distribution of mass is defined by the following metric in the Moffat theory[13]:

$$ds^2=\left[\frac{\left(1+\dfrac{m}{2r}\right)^8+\tau\left(\dfrac{R_s}{r}\right)^4}{\left(1+\dfrac{m}{2r}\right)^8}\right]\frac{\left(1-\dfrac{m}{2r}\right)^2}{\left(1+\dfrac{m}{2r}\right)^2}c^2dt^2 - \left(1+\frac{m}{2r}\right)^4(dx^2+dy^2+dz^2) \tag{A1}$$

where ds is the differential distance between two "world points" in the space/time continuum and the remaining variables are defined as for Eq. (4). When the parameter τ is set equal to zero, the metric of Eq. (A1) reduces to the Schwarzschild solution of the Einstein field equations. In the solar system, the parameter R_s is the mean radius of the sun.

For purposes of describing the paths of planets, spacecraft, and light rays in the solar system, only the post-Newtonian terms in Eq. (A1) are important. These terms can be isolated by expanding the time part of the metric to second order in m/r and the space part of the metric to first order in m/r. The metric that results is sufficient to represent the Moffat theory at the post-Newtonian level, and with $\tau=0$ it represents the Einstein theory as well. It can be generalized further to include a broader class of gravitational theories by adding the parameters β, γ, and α_1.[15] The final form of the metric used for Starprobe studies is given by the expression

$$\mathrm{d}s^2=\Big\{1-2\left(\frac{m}{r}\right)+2\beta\left(\frac{m}{r}\right)^2+\alpha_1\left(\frac{w^2}{c^2}\right)\left(\frac{m}{r}\right)$$
$$+\tau\left(\frac{R_s}{r}\right)^4\left[1-6\left(\frac{m}{r}\right)+19\left(\frac{m}{r}\right)^2\right]\Big\}c^2\mathrm{d}t^2$$
$$+\alpha_1\left(\frac{m}{r}\right)(w_x\mathrm{d}x+w_y\mathrm{d}y+w_z\mathrm{d}z)\,\mathrm{d}t$$
$$-\left[1+2\gamma\left(\frac{m}{r}\right)\right](\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2) \tag{A2}$$

where w_x, w_y, and w_z are the components of the vector w and w is its magnitude. The vector is as defined for Eq. (4).

Equations of Motion

In a metric theory of gravitation, such as general relativity, the equations of motion for freely falling particles are given by the equations of the geodesic path in the space/time continuum defined by the metric. The equations for this four-dimensional path can be found by the usual methods of Riemannian geometry, but it is easier in the post-Newtonian approximation to make use of the fact that the integral $\int(\mathrm{d}s/\mathrm{d}t)\,\mathrm{d}t$ is stationary along the path and to define a Lagrangian L for the motion by

$$\frac{L}{c^2}=1-\frac{1}{c}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right) \tag{A3}$$

The equations of motion follow by means of the Euler-Lagrange equations. But first, in forming the Lagrangian of Eq. (A3) from Eq. (A2), we employ the binomial expansion; and it is necessary to decide on where to truncate the infinite series. The Starprobe spacecraft will approach the sun at about four solar radii, and hence $m/r<6\times10^{-7}$ and $v^2/c^2\sim2m/r$, where v is the magnitude of $\dot{r}$. As a result of the Viking time-delay experiment, the parameter τ is limited in magnitude to 1.4×10^{-7} (see Ref. 18), and therefore $\tau(R_s/r)^4<6\times10^{-10}$. Also $w\sim v$. For Starprobe, it is reasonable to keep terms of second order in m/r, v^2/c^2, and w^2/c^2 and to keep products of τ with these three quantities. If all terms of higher order are neglected, then the Lagrangian of Eq. (A3) is given by

$$L=\frac{1}{2}v^2+\frac{\mu}{r}-\frac{1}{2}(2\beta-1)\left(\frac{m}{r}\right)\left(\frac{\mu}{r}\right)$$
$$+\frac{1}{2}(2\gamma+1)\left(\frac{m}{r}\right)v^2+\frac{1}{8}\left(\frac{v^4}{c^2}\right)-\frac{1}{2}\alpha_1\left(\frac{w^2}{c^2}+\frac{w\cdot\dot{r}}{c^2}\right)\left(\frac{\mu}{r}\right)$$
$$-\frac{1}{2}\tau\left(\frac{R_s}{r}\right)^4\left[c^2-5\left(\frac{\mu}{r}\right)+\left(\frac{1}{2}\right)v^2\right] \tag{A4}$$

The Euler-Lagrange equation in vector form is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{r}}\right)=\frac{\partial L}{\partial r} \tag{A5}$$

Substituting Eq. (A4) into Eq. (A5) and carrying out the differentiations yields the equation of motion

$$\ddot{r}=-\left(\frac{\mu}{r^3}\right)r+\frac{m}{r^3}\Big\{\left[2(\beta+\gamma)\left(\frac{\mu}{r}\right)-\gamma(\dot{r}\cdot\dot{r})\right]r$$
$$+2(1+\gamma)[r\cdot\dot{r}]\dot{r}\Big\}+\alpha_1\left(\frac{m}{2r^3}\right)\Big\{[w\cdot\dot{r}]r-[r\cdot\dot{r}]w\Big\}$$
$$-\tau\left(\frac{R_s^4}{r^6}\right)\Big\{\left[-2c^2+13\left(\frac{\mu}{r}\right)-(\dot{r}\cdot\dot{r})\right]r+2[r\cdot\dot{r}]\dot{r}\Big\} \tag{A6}$$

where the accelerations in the post-Newtonian terms [i.e., those with a coefficient of $(2\gamma+1)(m/r)$, $0.5(\dot{r}\cdot\dot{r})/c^2$, or $0.5\ \tau\ (R_s/r)^4$] have been replaced by their Newtonian expression

$$\ddot{r}_{\mathrm{Newton}}=-\mu r/r^3 \tag{A7}$$

and μ has been redefined as $(1-0.5\alpha_1w^2/c^2)$ times the μ that appears in Eq. (A4).

Gravitational Time Delay

If a light ray or radio signal is sent through the solar system, the time of propagation of the signal between two points in space will be delayed by the gravitational field of the sun. The light ray will propagate along a null geodesic ($\mathrm{d}s\equiv0$) and, according to Eq. (A2), the coordinate velocity v of the light ray will, to the first order, be determined by

$$\left[1+2\gamma\left(\frac{m}{r}\right)\right]v^2=\left[1-2\left(\frac{m}{r}\right)+\tau\left(\frac{R_s}{r}\right)^4\right]c^2 \tag{A8}$$

Here r is the coordinate distance between the sun and a point on the path of the light ray. Then, if the derivation of the signal delay is restricted to first-order terms, the total propagation time Δt for the signal can be obtained by integrating c/v along a straight line path between the two points in space separated by coordinate distance ρ.

The value of Δt for the straight path between the Earth and Starprobe is given by the integral

$$c\Delta t=\int_0^\rho\left(\frac{c}{v}\right)\mathrm{d}\rho \tag{A9}$$

where, from Eq. (A8), c/v is given to first order by the truncated series

$$\frac{c}{v}=1+(1+\gamma)\frac{m}{r}-\frac{1}{2}\tau\left(\frac{R_s}{r}\right)^4 \tag{A10}$$

Two integrals, $\int\mathrm{d}\rho/r$ and $\int\mathrm{d}\rho/r^4$, are needed. The easiest evaluation of these integrals is obtained by transforming to the angle variable θ_2 by means of the two identities

$$\rho=\frac{r_E\sin\theta_2}{\sin(\theta_1+\theta_2)} \tag{A11}$$

and

$$r=\frac{r_E\sin\theta_1}{\sin(\theta_1+\theta_2)} \tag{A12}$$

The variables r_E, θ_1, and θ_2 were defined earlier following Eq. (6). Note that the angle θ_1 and the distance r_E remain constant as r, ρ, and θ_2 vary over the path of the light ray. Thus, the derivative of Eq. (A11) yields

$$\mathrm{d}\rho=\frac{r_E\sin\theta_1}{\sin^2(\theta_1+\theta_2)}\mathrm{d}\theta_2 \tag{A13}$$

and

$$\int_0^\rho\frac{\mathrm{d}\rho}{r}=\int_0^{\theta_2}\frac{\mathrm{d}\theta_2}{\sin(\theta_1+\theta_2)} \tag{A14}$$

or

$$\int_0^\rho\frac{\mathrm{d}\rho}{r}=\ln\left[\tan\frac{1}{2}(\theta_1+\theta_2)\tan^{-1}\left(\frac{\theta_1}{2}\right)\right] \tag{A15}$$

Equation (A15) can be put in a more standard form by using relations between the sides and angles of the Earth/sun/spacecraft triangle, to obtain

$$\int_0^\rho\frac{\mathrm{d}\rho}{r}=\ln\left[\frac{r_E+r_P+\rho}{r_E+r_P-\rho}\right] \tag{A16}$$

where r_p is as defined for Eq. (6). The physical reality and significance of this excess time delay in a radar ranging signal was first recognized by Shapiro,[37] who derived an expression equivalent to Eq. (A16) for the Schwarzschild metric.

The decrease in the delay in the Moffat theory is obtained from the integral

$$\int_0^{\rho} \frac{d\rho}{r^4} = \int_0^{\theta_2} \frac{\sin^2(\theta_1+\theta_2)}{r_E^3 \sin^3\theta_1} d\theta_2 \tag{A17}$$

or

$$\int_0^{\rho} \frac{d\rho}{r^4} = \frac{1}{2r_E^3 \sin^3\theta_1}\left[\theta_2 - \frac{1}{2}\sin 2(\theta_1+\theta_2) + \frac{1}{2}\sin 2\theta_1\right] \tag{A18}$$

The final expression [see Eq. (6)] for the coordinate time of propagation of a signal between the Earth and Starprobe is obtained by combining Eqs. (A9), (A10), (A16), and (A18).

Acknowledgments

The research described in this paper was carried out by the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration. The authors thank E.L. Lau for checking our results with her independent covariance analysis software.

References

[1]Anderson, J.D. et al., "Tests of General Relativity Using Astrometric and Radio Metric Observations of the Planets," *Acta Astronautica,* Vol. 5, 1978, pp. 43-61.

[2]Dicke, R.H., "The Sun's Rotation and Relativity," *Nature,* Vol. 202, May 1964, pp. 432-435.

[3]Dicke, R.H. and Goldenberg, H.M., "The Oblateness of the Sun," *Astrophysical Journal,* Suppl. 241, Vol. 27, 1974, pp. 131-182.

[4]Hill, H.A. and Stebbins, R.T., "The Intrinsic Visual Oblateness of the Sun," *Astrophysical Journal,* Vol. 200, Sept. 1975, pp. 471-483.

[5]Roxburgh, I.W., "The Importance of Determining the Solar Quadrupole Moment," *A Close-Up of the Sun,* edited by M. Neugebauer and R.W. Davies, NASA/JPL Doc. 78-70, 1978, pp. 11-18.

[6]Randolph, J.E., "Starprobe: The Mission and System Options," AIAA Paper 82-0041, Jan. 1982.

[7]Anderson, J.D., Colombo, G., Friedman, L.D., and Lau, E.L., "An Arrow to the Sun," *Gravitazione Sperimentale,* Accademia Nazionale dei Lincei, Rome, 1977, pp. 393-422.

[8]Reasenberg, R.D. and Shapiro, I.I., "Possible Measurement of J_2 with a Solar Probe," *A Close-Up of the Sun,* edited by M. Neugebauer and R.W. Davies, NASA/JPL Doc. 78-70, 1978, pp. 19-41.

[9]Anderson, J.D. and Lau, E.L., "Gravitational Experiments on Solar Probe," *A Close-Up of the Sun,* edited by M. Neugebauer and R.W. Davies, NASA/JPL Doc. 78-70, 1978, pp. 42-57.

[10]Nordtvedt, K., "Possible Measurements of J_2 and the Sun's Angular Momentum with the Solar Probe," *A Close-Up of the Sun,* edited by M. Neugebauer and R.W. Davies, NASA/JPL Doc. 78-70, 1978, pp. 58-59.

[11]Carta, D.G., "Navigation Simulation Results for Drag-Free Solar Probe Mission," JPL Internal Doc. IOM 314.7-336, Aug. 1979.

[12]Mease, K.D., Wood, L.J., Bergam, M.J., and White, L.K., "Estimation of Solar Gravitational Harmonics with Starprobe Radiometric Tracking Data," *Journal of the Astronautical Sciences,* Vol. 31, Jan.-March 1983, pp. 3-22.

[13]Moffat, J.W., "Consequences of a New Experimental Determination of the Quadrupole Moment of the Sun for Gravitation Theory," *Physical Review Letters,* Vol. 50, No. 10, March 1983, pp. 709-712.

[14]Dicke, R.H., "Evidence for a Solar Distortion Rotating with a Period of 12.2 Days," *Solar Physics,* Vol. 47, 1976, pp. 475-515.

[15]Will, C.M., *Theory and Experiment in Gravitational Physics,* Cambridge University Press, New York, 1981.

[16]Reasenberg, R.D. et al., "Viking Relativity Experiment: Verification of Signal Retardation by Solar Gravity," *Astrophysical Journal,* Vol. 234, Dec. 1979, pp. L219-L221.

[17]Warburton, R.J. and Goodkind, J.M., "Search for Evidence of a Preferred Reference Frame," *Astrophysical Journal,* Vol. 208, 1976, pp. 881-886.

[18]Moffat, J.W., "Experimental Tests of a New Theory of Gravitation," *Canadian Journal of Physics,* Vol. 59, No. 2, 1981, pp. 283-288. Erratum, Vol. 59, 1981, p. 1289.

[19]Weinberg, S., *Gravitation and Cosmology,* John Wiley & Sons, New York, 1972, p. 232.

[20]Cheng, E.S., Saulson, P.R., Wilkinson, D.T., and Corey, B.E., "Large-Scale Anisotropy in the 2.7K Radiation," *Astrophysical Journal,* Vol. 232, Sept. 1979, pp. L139-L143.

[21]Deubner, F.L., Ulrich, R.K., and Rhodes, E.D. Jr., "Solar p-Mode Oscillations as a Tracer of Radial Differential Rotation," *Astronomy and Astrophysics,* Vol. 72, 1979, pp. 177-185.

[22]Brooks, J.R., Isaak, G.R., and van der Raay, H.B., "Observations of Free Oscillations of the Sun," *Nature,* Vol. 259, 1976, pp. 92-95.

[23]Severny, A.B., Kotov, V.A., and Tsap, T.T., "Observations of Solar Pulsations," *Nature,* Vol. 259, 1976, pp. 87-89.

[24]Douglass, D.H., "Detection of Gravitational Radiation and Oscillations of the Sun via Doppler Tracking of Spacecraft," *A Close-Up of the Sun,* edited by M. Neugebauer and R.W. Davies, NASA/JPL Doc. 78-70, 1978, pp. 498-517.

[25]Johnson, W.W., Winget, D.H., Douglass, D.H., and Van Horn, H.M., "Time-Varying Gravitational Multipole Moments Corresponding to Nonradial Solar Oscillations," *Nonradial and Nonlinear Stellar Pulsation,* edited by H.A. Hill and W.A. Dziembowski, Springer-Verlag, New York, 1980, pp. 358-368.

[26]Christensen-Dalsgaard, J. and Gough, D.O., "Perturbations in Gravitational Potential Associated with Solar Oscillations," *Nonradial and Nonlinear Stellar Pulsation,* edited by H.A. Hill and W.A. Dziembowski, Springer-Verlag, New York, 1980, pp. 369-380.

[27]Dicke, R.H., "Seismology and Geodesy of the Sun: Solar Geodesy," *Proceedings of the National Academy of Sciences* (USA), Vol. 78, No. 3, 1981, pp. 1309-1312.

[28]Ulrich, R.K. and Hawkins, G.W., "The Solar Gravitational Figure—J_2 and J_4," JPL Internal Doc. IOM 312/80.5-763, Att. 4-1, 1980.

[29]Everitt, C.W.F. and DeBra, D.B., "Comments on the Drag-Free Control of a Solar Probe Relativity Mission," *A Close-up of the Sun,* edited by M. Neugebauer and R.W. Davies, NASA/JPL Doc. 78-70, 1978, pp. 60-68.

[30]Armstrong, J.W., Woo, R., and Koerner, M., "Starprobe: Coronal Plasma Turbulence Effects on Tracking and Telemetry," AIAA Paper 82-0044, Jan. 1982.

[31]Bierman, G.J., *Factorization Methods for Discrete Sequential Estimation,* Academic Press, New York, 1977.

[32]*Solar Probe Advanced Technical Development Midyear Status Report,* NASA/JPL Internal Doc. 715-44, June 1980.

[33]Jazwinski, A.H., *Stochastic Processes and Filtering Theory,* Academic Press, New York, 1970, pp. 281-286.

[34]Lorell, J., "Estimation of Gravity Field Harmonics in the Presence of Spin-Axis Direction Error Using Radio Tracking Data," *Journal of the Astronautical Sciences,* Vol. 20, No. 1, July-Aug. 1972, pp. 44-54.

[35]Williams, J.G. et al., "New Test of the Equivalence Principle from Lunar Laser Ranging," *Physical Review Letters,* Vol. 36, 1976, pp. 551-554.

[36]Shapiro, I.I., Counselman, C.C. III, and King, R.W., "Verification of the Principle of Equivalence for Massive Bodies," *Physical Review Letters,* Vol. 36, 1976, pp. 555-558.

[37]Shapiro, I.I., "Fourth Test of General Relativity," *Physical Review Letters,* Vol. 13, 1964, pp. 789-791.

[38]Hill, H.A., Bos, R.J., and Goode, P.R., "Preliminary Determination of the Sun's Gravitational Quadrupole Moment from Rotational Splitting of Global Oscillations and Its Relevance to Tests of General Relativity," *Physical Review Letters,* Vol. 49, 1982, pp. 1794-1797.